Teacher notes
Topic B

How do we know stars are made of gas?

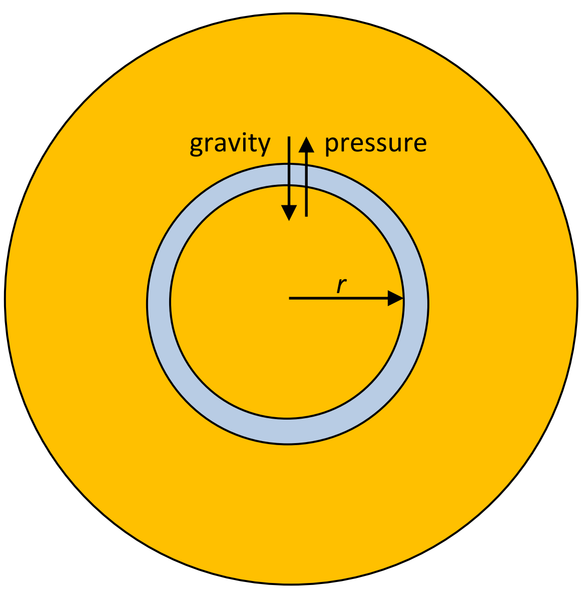
This requires lots of math and lots of order of magnitude estimates. Not for everyone.

Apart from exotic stars like white dwarfs and neutron stars, stars are made of gas. But how do we know this?

The defining characteristic of a gas is that the random kinetic energy of the molecules is much larger than the intermolecular potential energy (which is zero for an *ideal* gas). So consider a star of density ρ that is made of atoms of nucleon number A and proton number Z at some temperature T. (Density and temperature vary of course in the interior of a star but this is a rough order of magnitude estimate so we will assume they are constant.)

The number of atoms per unit volume (the number density) is $\dfrac{m}{\rho}$ where m is the mass of an atom and ρ is the density. So, the typical separation of these atoms is $d=(\dfrac{m}{\rho})^{\frac{1}{3}}$. The typical electric potential energy of a pair of atoms is then $E=\dfrac{kZ^2e^2}{d}=kZ^2e^2\left(\dfrac{\rho}{m}\right)^{\frac{1}{3}}=kZ^2e^2\left(\dfrac{M}{\frac{4\pi}{3}R^3m}\right)^{\frac{1}{3}}=\dfrac{kZ^2e^2}{R}\left(\dfrac{M}{\frac{4\pi}{3}m}\right)^{\frac{1}{3}}$.

Consider a shell of radius r and thickness Δr.



The difference in pressure between the inner and outer surfaces of the shell is ΔP. The force pushing the shell outwards is then $\Delta P \times 4\pi r^2$. The gravitational force on the shell is $\frac{G}{r^2} \times \frac{Mr^3}{R^3} \times 4\pi r^2 \rho$ (only the mass within the shell attracts the shell). Equating the two forces for equilibrium gives

$$\Delta P \times 4\pi r^2 = \frac{G}{r^2} \times \frac{Mr^3}{R^3} \times 4\pi r^2 \rho \Delta r$$
$$\frac{\Delta P}{\Delta r} = \frac{GMr}{R^3}\rho$$

Integrating

$$\int_0^P dP = -\int_0^R \frac{GMr}{R^3}\rho dr$$

gives

$$P = \frac{GM}{R^3}\frac{R^2}{2}\rho = \frac{GM}{R}\frac{1}{2}\frac{M}{\frac{4\pi}{3}R^3} = \frac{3}{8\pi}\frac{GM^2}{R^4}$$ (assuming zero pressure outside the star).

This gives an estimate for the pressure at the centre of the star.

Inserting into the ideal gas law we find

$$PV = Nk_B T$$
$$T = \frac{PV}{Nk_B} = \frac{\frac{3}{8\pi}\frac{GM^2}{R^4}V}{\frac{M}{m}k_B} = \frac{\frac{3}{8\pi} \times \frac{GM^2}{R^4} \times \frac{4\pi R^3}{3}}{\frac{M}{m}k_B} = \frac{1}{2}\frac{GMm}{Rk_B}$$

This gives an estimate for the temperature at the centre of the star. For the Sun this evaluates to

$$T_{\odot} \approx \frac{1}{2} \times \frac{6.67 \times 10^{-11} \times 2 \times 10^{30} \times 1.67 \times 10^{-27}}{7 \times 10^8 \times 1.38 \times 10^{-23}} \approx 12 \times 10^6\ \text{K}$$ which is not a bad estimate at all.

The random kinetic energy of 2 atoms is $E_K = 2 \times \frac{3}{2}k_B T = \frac{3}{2}\frac{GMm}{R}$.

We found earlier that the potential energy of the pair is

$$E_P = \frac{kZ^2e^2}{R}\left(\frac{M}{\frac{4\pi}{3}m}\right)^{\frac{1}{3}}$$

The ratio is

$$\frac{E_{\mathrm{P}}}{E_{\mathrm{K}}}=\frac{\dfrac{kZ^2e^2}{R}\left(\dfrac{M}{\frac{4\pi}{3}m}\right)^{\frac{1}{3}}}{\dfrac{3}{2}\dfrac{GMm}{R}}=\frac{2}{3}\left(\frac{3}{4\pi}\right)^{\frac{1}{3}}\frac{kZ^2e^2}{G}(Mm^2)^{-\frac{2}{3}}\approx 1.4\times 10^{-18}\times Z^2(Mm^2)^{-\frac{2}{3}}$$

For a one solar mass star with hydrogen, this ratio is

$$\frac{E_{\mathrm{P}}}{E_{\mathrm{K}}}\approx 1.4\times 10^{-18}\times 1^2\times(2\times 10^{30}\times(1.67\times 10^{-27})^2)^{-\frac{2}{3}}\approx 5\times 10^{-3}$$

and for a 10 solar mass star made out of iron this is

$$\frac{E_{\mathrm{P}}}{E_{\mathrm{K}}}\approx 1.4\times 10^{-18}\times 26^2\times(2\times 10^{31}\times(9.4\times 10^{-26})^2)^{-\frac{2}{3}}\approx 3\times 10^{-3}$$

In both cases $E_{\mathrm{P}} << E_{\mathrm{K}}$, consistent with the fact that the star is made out of gas!